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State-Identification Experiments in Finite Automata*

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Work done in the past on the subject of state-identification in finite automata has been limited to devising procedures for some special-purpose identification experiments, and to estimating the lengths of more general experiments. The purpose of the present paper is to complement this work by presenting procedures for solving general state-identification problems by experiments of various specified characteristics. Classifying identification experiments into preset or adaptive and into simple or multiple, the results contain procedures for identifying the initial state of an automaton by minimal, simple, preset or adaptive experiments, whenever simple, preset experimentation is realizable, and by multiple, preset or adaptive experiments, whenever simple experimentation is not realizable. Also, procedures are described for identifying the final state of an automaton by minimal or nonminimal, preset or adaptive experiments. Bounds associated with the various procedures are determined, and the applicability of the results to machine-identification problems is discussed.

I. INTRODUCTION

The problem of identifying the initial or final state of a finite automaton (or "machine") has been drawing little attention, despite its great significance in the theory of automata. To the author's knowledge, the only papers devoted to this subject are those by Moore (1956) and Ginsburg (1958), whose important contributions still leave a great number of questions unanswered. Before summarizing these contributions and outlining the objectives of the present paper, it is advantageous to discuss in detail the meaning and purpose of "identification experiments."

The two identification problems under investigation are the following:

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1. *The distinguishing problem:* It is known that a given machine M is in one of the states $\sigma_1, \sigma_2, \cdots, \sigma_q$; find the state.

2. *The homing problem:* It is known that a given machine M is in one of the states $\sigma_1, \sigma_2, \cdots, \sigma_q$; pass M into a known state.

In each of the above problems, M is assumed to be a "black box," with only the input and output terminals accessible to the experimenter. The state table (or diagram) of M is assumed to be available in a reduced form for the task of designing the identification experiments. These experiments are conducted by applying input sequences to M and observing corresponding output sequences; the solution is then arrived at on the basis of the applied and observed sequences. The "length" of an experiment is taken as the number of input symbols applied to M from the time the problem is posed to the time it is solved.

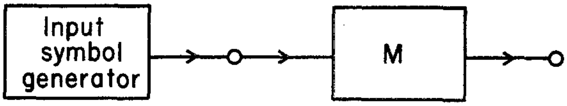
Identification experiments are divisible into the following categories:

(a) *Preset experiments:* The input sequence is completely designed in advance and is valid regardless of the initial state of M.

(b) *Adaptive experiments:* The input sequence is composed of a number of subsequences, each subsequence (except the first) designed on the basis of outputs yielded by preceding subsequences. The sequence, in general, depends on the initial state of M.

A preset experiment, as a rule, is easier to implement than an adaptive one, since it requires no decision-making before the experiment terminates. An adaptive experiment, on the other hand, requires a number of intermediate decisions before termination. Envisioning a human or a mechanical "input symbol generator" supplying M with the designed sequences, it can be seen that in the case of a preset experiment the generator should be capable of supplying a single sequence only. In the case of an adaptive experiment, the generator should be capable of generating a number of sequences, each sequence determined by information fed back from the output terminal. This interpretation of the two types of experiments is depicted by Fig. 1. Assuming some probability distribution over the states $\sigma_1, \sigma_2, \cdots, \sigma_q$, the advantage of an adaptive experiment is that it has a nonzero probability of being shorter than a preset experiment designed for the same problem. Also, in some problems, an adaptive experiment is easier to design than a preset one.

Identification experiments can also be classified according to the number of "copies" of M which they require (one machine is a copy of another if both machines are describable by the same state table, and if both are in the same state before the experiment commences):



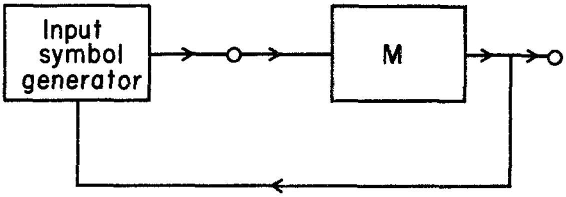


FIG. 1. (a) Preset experiment. (b) Adaptive experiment

(a) *Simple experiments:* Only one copy of M is required.
(b) *Multiple experiments:* More than one copy of M is required.

Clearly, a simple experiment is preferable to a multiple one, inasmuch as most machines encountered in practice are available in one copy only.

A simple, preset experiment is called *minimal* if it is the shortest one to yield the desired solution. A simple, adaptive experiment is called minimal if its length does not exceed the length of the minimal, simple, preset experiment designed for the same problem. Simple experiments which are not designed by a procedure guaranteeing their minimality (although they may, in some specific situations prove to be minimal), are called *regular*.

The set of states $\{\sigma_1, \sigma_2, \cdots, \sigma_q\}$, one of which is the initial state of M, will be called the *admissible set* and denoted by $\mathcal{A}$. The elements of $\mathcal{A}$ will be called *admissible states*. The total number of states in M will be denoted by n.

Moore's paper presents a procedure for solving the distinguishing problem, with $q = 2$, by a minimal, simple, preset experiment. It also presents a procedure for solving the homing problem, with $q = n$, by a regular, simple, adaptive experiment. Ginsburg's paper presents bounds on the length of a minimal, simple, preset experiment for solving

the homing problem. In the present paper we shall complement these results by describing:

(1) A procedure for solving the distinguishing problem (for any q) by a minimal, simple, preset experiment, whenever a solution by simple, preset experimentation is at all possible.

(2) A procedure for solving the distinguishing problem (for any q) by a minimal, simple, adaptive experiment, whenever a solution by simple, preset experimentation is at all possible.

(3) A procedure for solving the distinguishing problem (for any q) by a multiple, preset experiment, whenever a solution by simple experimentation is impossible.

(4) A procedure for solving the distinguishing problem (for any q) by a multiple, adaptive experiment, whenever a solution by simple experimentation is impossible.

(5) A procedure for solving the homing problem (for any q) by a minimal, simple, preset experiment.

(6) A procedure for solving the homing problem (for any q) by a minimal, simple, adaptive experiment.

(7) A procedure for solving the homing problem (for any q) by a regular, simple, preset experiment.

(8) A procedure for solving the homing problem (for any q) by a regular, simple, adaptive experiment.

In addition, we shall present some bounds associated with the various procedures, and discuss the applicability of the results to the problem of machine-identification.

II. SUCCESSOR TREES

Before describing any of the procedures, it is useful to introduce a number of definitions.

An S_i-*set* is an unordered set of states of M, which may contain any number of identical states. An S-*group*, written as $\{S_1\}, \{S_2\}, \cdots, \{S_r\}$, is an unordered set of S_i-sets, not necessarily disjoint. An S_i-set containing a single element is called *simple*; an S-group in which all S_i-sets are simple is likewise referred to as simple. An S_i-set which contains two or more identical states is called *multiple*. An S_i-set in which all states are identical is called *homogeneous* (a simple set being a special case); an S-group in which all S_i-sets are homogeneous is likewise referred to as homogeneous.

An input sequence composed of the symbols η_{i_1}, followed by $\eta_{i_2}, \cdots$, followed by η_{i_l}, will be written as $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$. Given an S-group S and an input sequence $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$, the $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$*-successor of* S is an S-set derivable from S as follows: (a) Partition each S_i-set of S into the subsets $\{S_{i_1}\}, \{S_{i_2}\}, \cdots, \{S_{ip_i}\}$, such that two states in $\{S_i\}$ belong to the same subset $\{S_{i_j}\}$ if and only if they yield identical responses to $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$. (b) Replace every state σ_k in every subset $\{S_{i_j}\}$ by the state σ_k' into which σ_k passes when $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ is applied to M; denote the resulting subset by $\{S'_{i_j}\}$. The set of subsets $\{S'_{i_j}\}$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, p_i$, constitutes the $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$-successor of S.

The *successor tree* is a structure composed of *branches* arranged in successive *levels*, the uppermost level being the "first" level, the next to uppermost being the "second" level, and so forth. Every branch is associated with an S-group; a branch associated with the S-group S is referred to as the branch "S." A branch is either *terminal*, when it does not generate any next-level branches, or *nonterminal*, when it generates m next-level branches, one branch for each symbol in the input alphabet $\{\eta_1, \eta_2, \cdots, \eta_m\}$. Thus, every branch (excluding first-level branches) is associated with both an S-group and an input symbol η_i. For any given machine M and an admissible set $\mathcal{A}$, the successor tree can be constructed, level by level, according to the following rules:

(i) Let the first level consist of the single branch "$\mathcal{A}$."

(ii) Let branch "S" be a terminal branch if any of the following conditions exists:

(a) S contains any multiple S_i-sets.

(b) There is a branch "S" in a previously completed level.

(iii) If none of the conditions listed under (ii) exists, let branch "S" generate m next-level branches. Let the kth branch be "$S^{(k)}$," where $S^{(k)}$ is the η_k-successor of S.

(iv) Terminate construction when no new branches can be generated, or when any of the S-groups associated with the last completed level is simple.

Figure 2 shows the state diagram and Table I the state table for machine A. In the table, s_t, x_t, and y_t are the state, input symbol, and output symbol respectively, at time t. In the diagram, a branch pointing from state σ_i to state σ_j and labeled (η_k/ρ_l) indicates that the input symbol η_k causes state σ_i to pass into state σ_j, yielding the output symbol ρ_l. Both the tabular and diagramatic representations follow the model proposed by Mealy (1955), which was shown (by Cadden, 1959,

TABLE I

	s_{t+1}		y_t	
s_t \ x_t	α	β	α	β
1	1	4	0	1
2	1	5	0	1
3	5	1	0	1
4	3	4	1	1
5	2	5	1	1

and by Gill, 1960) to be as general as Moore's model. Figure 3 shows the successor tree for A and $\mathcal{A} = \{2, 3, 4, 5\}$. As an example where both rules (ii), (a) and (ii), (b) are invoked in the tree construction, Fig. 4 shows the successor tree for A and $\mathcal{A} = \{1, 2\}$.

The following theorem serves to show that the construction of a successor tree is a finite process:

THEOREM 1. *The number of levels in a successor tree for an n-state machine and an admissible set of size q cannot exceed* $(q - 1)n^q$.

PROOF. Consider an S-group $\{S_1\}, \{S_2\}, \cdots, \{S_r\}$, denoted by $S^{(r)}$ and associated with some branch in this tree. Let the number of states in $\{S_i\}$ be ν_i; we then have $\nu_1 + \nu_2 + \cdots + \nu_r = q$. The number of distinct S-groups which contain r S_i-sets such that $\{S_i\}$ contains ν_i states is at most $n^{\nu_1}n^{\nu_2} \cdots n^{\nu_r} = n^q$. Thus, after at most n^q branches, branch "$S^{(r)}$" must either lead to a terminal branch, by virtue of rule (ii), (a) or rule (ii), (b), or lead to a branch associated with an S-group, denoted by $S^{(r+1)}$, which contains at least $r + 1$ S_i-sets. By induction, after at most $(q - 1)n^q$, a branch containing a single S_i-set must either lead to a terminal branch, by virtue of rule (ii), (a) or rule (ii), (b), or lead to a branch containing q S_i-sets. Since a group containing q S_i-sets is necessarily simple, the latter branch must also be terminal, by virtue of rule (iv). The number of levels in the tree, therefore, cannot exceed $(q - 1)n^q$.

A tree path leading from branch "S" to a branch associated with the $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$-successor of S will be said to "describe" the sequence $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$. A *distinguishing path* will be defined as any path which leads from $\mathcal{A}$ to a branch associated with a simple group. A *distinguishing sequence* for M and $\mathcal{A}$ is any predetermined input sequence which, when applied to M, determines the admissible state in which M was prior to application.

THEOREM 2. *A distinguishing path in a successor tree for machine M and admissible set $\mathcal{A}$ describes a distinguishing sequence for M and $\mathcal{A}$.*

PROOF. Let σ_i and σ_j be two states in $\mathcal{A}$. By construction, if the input sequence $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ passes σ_i into σ_i' and σ_j into σ_j', and if σ_i' and σ_j' are in two separate S_i-sets in the $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$-successor of $\mathcal{A}$, then the responses of σ_i and σ_j to $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ are distinct. Thus, if $\mathcal{A} = \{\sigma_1, \sigma_2, \cdots, \sigma_q\}$ and if the $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$-successor of $\mathcal{A}$ has q simple S_i-sets (and hence constitutes a simple group), the responses of σ_1, $\sigma_2, \cdots, \sigma_q$ to $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ are distinct; the response of M to this sequence, then, uniquely determines the initial state. Since, by definition,

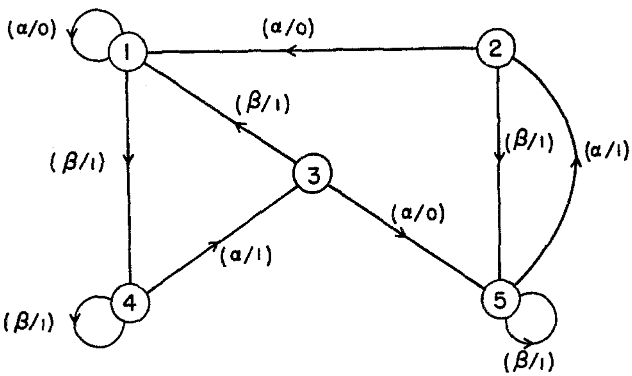


FIG. 2. Machine A

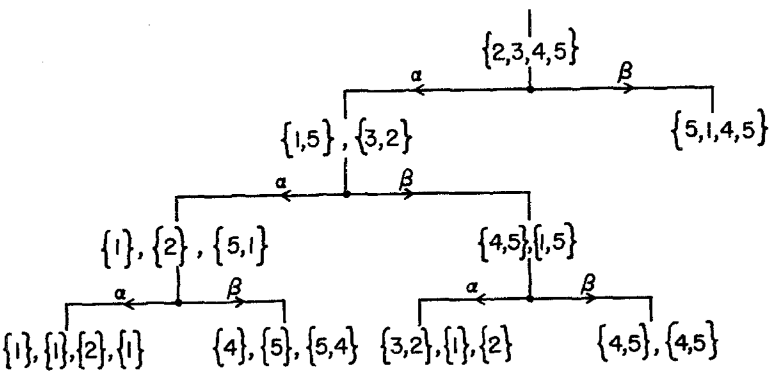


FIG. 3. Successor tree for A and $\mathcal{A} = \{2, 3, 4, 5\}$

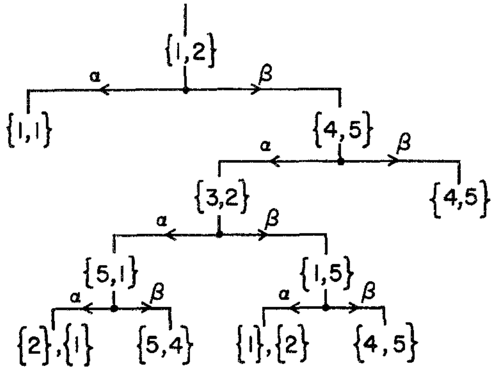


FIG. 4. Successor tree for A and $\mathcal{A} = \{1, 2\}$

an input sequence leading from $\mathcal{A}$ to a simple group corresponds to a distinguishing path, the theorem follows.

The shortest distinguishing sequence for machine M and admissible set $\mathcal{A}$, will be called the *minimal distinguishing sequence* for M and $\mathcal{A}$.

THEOREM 3. *The set of distinguishing paths in the successor tree for machine M and admissible set $\mathcal{A}$, describes all minimal distinguishing sequences for M and $\mathcal{A}$, and no distinguishing sequences other than minimal.*

PROOF. Consider an "extended" successor tree, constructed according to rules (i), (iii), and (iv) only. Paths appearing in the extended tree but not in the original one are those generated due to the omission of rule (ii). Now, an S-group containing a multiple S_i-set contains at least one S_i-set in which there are two or more identical states. Since the responses of these states to any input sequence are, clearly, identical, all successors to this S-group must also contain a multiple S_i-set. Thus, paths appearing in the extended but not in the original tree, due to the omission of rule (ii), (a), cannot be distinguishing paths. Consider now two identical S-groups S, associated with branches b_u and b_v, in the tree levels u and v, respectively, where $u < v$. Since b_u and b_v generate identical subtrees, if a distinguishing path traverses b_v, one also traverses b_u; since $u < v$, the former path is longer than the latter and hence cannot be minimal. Thus, paths appearing in the extended but not in the original tree, due to the omission of rule (ii), (b), cannot describe minimal distinguishing sequences. Consequently, since the extended tree describes all minimal distinguishing sequences for M and

$\mathfrak{A}$, the original tree must also describe them. Moreover, by virtue of rule (iv), the original tree cannot contain any distinguishing paths of distinct lengths, which proves the theorem.

III. SIMPLE DISTINGUISHING EXPERIMENTS

The results obtained in the preceding section suggest the following procedure for solving the distinguishing problem by a simple preset experiment:

(1) Construct the successor tree for the given machine M and $\mathfrak{A} = \{\sigma_1, \sigma_2, \cdots, \sigma_q\}$.

(2) Choose any of the distinguishing paths exhibited by the tree, say E, and list the responses of $\sigma_1, \sigma_2, \cdots, \sigma_q$ to E. If no distinguishing path is exhibited, solution is impossible by simple preset experimentation.

(3) Apply E to M and determine the initial state of M by comparing the response with the list compiled in step (2).

The above procedure constitutes a solution to the distinguishing problem by a minimal, simple, preset experiment, whenever simple preset experimentation is at all possible. From Theorem 1 it follows that the length of such an experiment never exceeds $(q - 1)n^q$.

As an example, the minimal distinguishing sequence for A and $\mathfrak{A} = \{2, 3, 4, 5\}$ is seen from Fig. 3 to be $\alpha\alpha\alpha$ (leading to the simple group $\{1\}, \{1\}, \{2\}, \{1\}$). The response of A to $\alpha\alpha\alpha$ when the initial state is 2, 3, 4, 5, is 000, 010, 101, 100, respectively. The response to $\alpha\alpha\alpha$, then, uniquely determines the initial state. Figure 4 shows that the minimal distinguishing sequences for A and $\mathfrak{A} = \{1, 2\}$ are $\beta\alpha\alpha\alpha$ and $\beta\alpha\beta\alpha$. When $q = 2$, as is the case in the example of Fig. 4, the method proposed in Moore's paper for distinguishing between a pair of states is simpler than the successor tree method. However, the tree method is still useful when *all* minimal distinguishing sequences are desired.

Consider a distinguishing path, broken up into segments according to the following algorithm: Calling $\mathfrak{A}$ "$S^{(1)}$", let the kth segment be the subpath leading from "$S^{(k)}$" to the first S-group, denoted by $S^{(k+1)}$, which contains at least one more simple S_i-set than $S^{(k)}$. Since the number of S_i-sets in the last S-group is q, the total number of segments cannot exceed $q - 1$. Denoting the subsequence described by the kth segment by E_k, an adaptive experiment can be conducted according to the following program: Starting with $k = 1$, apply E_k to M and observe the response. If the response is attributable to a single state in

$\mathfrak{A}$, this state is the sought initial state; if not, increment k by 1 and repeat.

This experiment is seen to be a minimal, simple, adaptive experiment for solving the distinguishing problem, and it can always be conducted if solution by simple preset experimentation is at all possible.[1] The experiment may terminate after any E_k, depending on the initial state σ_i of M (which is, of course, unknown at the outset). If $S^{(k+1)}$ contains the simple S_i-set $\{\sigma_i'\}$, and if applying the input sequence $E_1E_2 \cdots E_k$ passes σ_i into σ_i', then the response of M to $E_1E_2 \cdots E_k$ is uniquely attributable to σ_i and no additional experimentation is necessary. Thus, although at the outset there is no way of estimating to what extent (if at all) the adaptive experiment is shorter than the preset one, there is always a nonzero a priori probability that it will be shorter.

As an example, the problem of Fig. 3 can be solved by an adaptive experiment, by segmenting the minimal sequence $\alpha\alpha\alpha$ into $\alpha\alpha$ and α. If the initial state of A is 2, the response to the first subsequence $\alpha\alpha$ is 00, which can be attributed to the admissible state 2 only. The distinguishing experiment, in this case, would terminate after the application of two input symbols.

IV. MULTIPLE DISTINGUISHING EXPERIMENTS

In cases where the successor tree contains no distinguishing paths [this happens when all paths terminate by virtue of rule (ii)], one might resort to multiple experimentation in order to identify the initial state. Before describing the design procedure for these cases, it is useful to state the following theorem, which is proved in Ginsburg's paper: *In any set of w distinguishable states belonging to an n-state machine, there is at least one pair of states distinguishable by an input sequence of length $n - w + 1$ or less.*

Let an *admissible group* $\mathfrak{A}^{(k)}$ be an unordered set of admissible sets $\{\mathfrak{A}_1^{(k)}, \mathfrak{A}_2^{(k)}, \cdots, \mathfrak{A}_r^{(k)}\}$, with $\mathfrak{A}^{(1)} = \{\mathfrak{A}\}$. $\mathfrak{A}^{(k+1)}$ is obtained from $\mathfrak{A}^{(k)}$ as follows: If $\mathfrak{A}_i^{(k)}$ contains w_i states, it contains at least one pair of states,

[1] The author is indebted to the unknown reviewer of this paper for pointing out that simple, adaptive distinguishing experiments may exist in cases where simple, preset experiments are not realizable. These experiments correspond to paths in the successor tree which traverse multiple S_i sets, but in which such sets are eliminable by virtue of previously observed responses. The simple, adaptive distinguishing experiments proposed here are restricted to those obtainable by early terminations of simple, preset experiments, and hence are constructible only when simple, preset experimentation is at all possible.

say σ_i and σ_j, which are distinguishable by an input sequence of length $n - w_i + 1$ or less. This sequence, denoted by $E_i^{(k)}$, can be found by Moore's method or through the successor tree. Partition $\mathcal{A}_i^{(k)}$ into the subsets $\mathcal{A}_{i_1}^{(k)}, \mathcal{A}_{i_2}^{(k)}, \cdots, \mathcal{A}_{i_{p_i}}^{(k)}$, such that two states of $\mathcal{A}_i^{(k)}$ are in the same subset if and only if they yield identical responses to $E_i^{(k)}$. There will be at least two such subsets, each containing at most $w_i - 1$ states. The admissible group $\mathcal{A}^{(k+1)}$, then, consists of those $\mathcal{A}_{i_j}^{(k)}$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, p_i$, which contain more than one state. Following this definition, all admissible groups for any given M and $\mathcal{A}$ can be determined in a recursive manner. Since successive admissible groups represent successive refinements of $\mathcal{A}$, the described process will terminate when $k \leqq q - 1$.

The multiple experiment can now be designed as follows: Suppose the number of distinguishing sequences determined in the process described in the preceding paragraph is C. Apply these sequences to C different copies of M—one sequence per copy. Inspect the response to $E_1^{(1)}$; this response can be attributed to only one $\mathcal{A}_i^{(2)}$, say $\mathcal{A}_{i_2}^{(2)}$. Inspect the response to $E_{i_2}^{(2)}$; this response can be attributed to only one $\mathcal{A}_i^{(3)}$, say $\mathcal{A}_{i_3}^{(3)}$. In general: Inspect the response to $E_{i_k}^{(k)}$; this response can be attributed to only one $\mathcal{A}_i^{(k+1)}$, say $\mathcal{A}_{i_{k+1}}^{(k+1)}$; increment k by 1 and repeat. Continue the inspection until an $\mathcal{A}_i^{(k)}$ is encountered which contains a single state (this must occur for $k \leqq q - 1$); the single state is the initial state sought.

The multiple, preset experiment described above can be displayed diagramatically by means of a "multiple experiment tree." This is done in Fig. 5 for machine A and $\mathcal{A} = \{1, 2, 3, 4, 5\}$. As can be verified from the successor tree for A and $\mathcal{A}$, the distinguishing problem in this case cannot be solved by a simple experiment. In the multiple experiment tree, each nonterminal branch represents a different copy of M. Indicated against the input terminal of each copy are the $\mathcal{A}_i^{(k)}$ set, the $E_i^{(k)}$ sequence, and the pair of states (σ_i, σ_j) for which $E_i^{(k)}$ is the minimal distinguishing sequence. σ_i and σ_j are selected such that their minimal distinguishing sequence is shorter than, or as short as, the distinguishing sequence for any other pair in $\mathcal{A}$ (this choice guarantees the minimality of the individual sequences, but not necessarily the minimality of the total experiment). Indicated against the output terminal of each copy are the responses attributable to states in $\mathcal{A}_i^{(k)}$, distinct responses generating distinct next-level branches, associated with disjoint subsets of $\mathcal{A}_i^{(k)}$. A tree path starting at $\mathcal{A}_1^{(1)}$ and ending at any single-state set $\{\sigma_i\}$, indicates the sequence of response inspections conducted when the

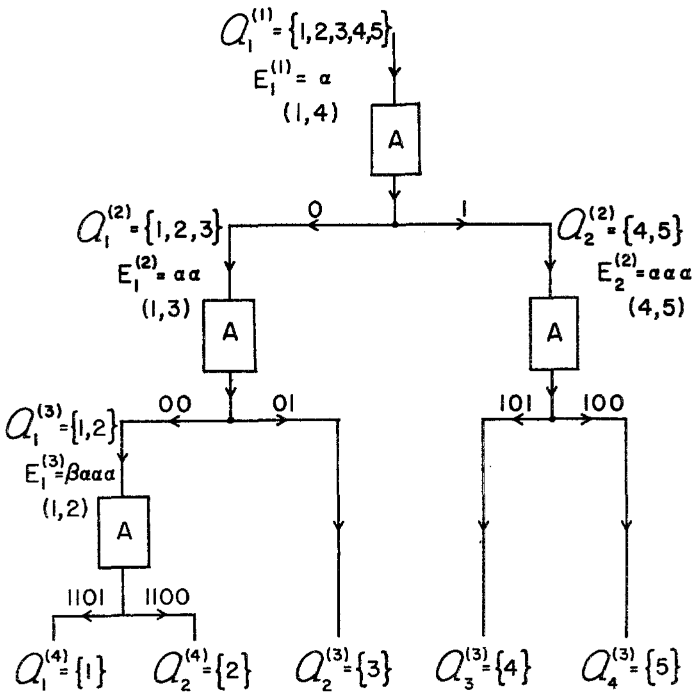


FIG. 5. Multiple experiment tree for A and $\mathcal{A} = \{1, 2, 3, 4, 5\}$

initial state of M is σ_i. For example, when the initial state of A is 3, $E_1^{(1)}$ (and its response 0) lead to $E_1^{(2)}$ (and its response 01), which lead to the terminal set $\mathcal{A}_2^{(3)} = \{3\}$ and hence to the identification of the initial state.

It can be noticed that each node in the kth level of the multiple experiment tree "splits" the $\mathcal{A}_i^{(k)}$ associated with the incoming branch into two or more disjoint subsets. Since the number of splitting operations required to split a q-state set into q single-state subsets cannot exceed $q - 1$, the tree contains at most $q - 1$ nodes. Since each node is an output terminal of M, the tree contains at most $q - 1$ copies of M. Consequently, we have the following theorem:

THEOREM 4. *The initial state of machine M having an admissible set of size q can always be identified by a preset experiment requiring C copies of M, where $C \leqq q - 1$.*

Table II is the state table of an n-state machine for which the upper

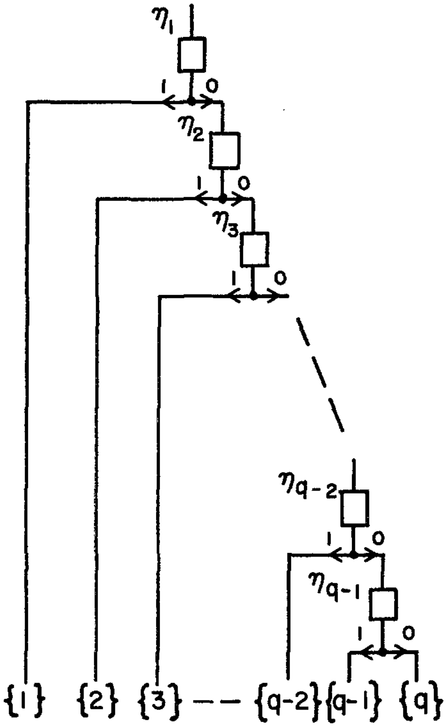


FIG. 6. Multiple experiment tree for machine of Table II

TABLE II

	s_{t+1}						y_t					
$s_t \backslash x_t$	η_1	η_2	η_3	$\cdots$	η_{n-2}	η_{n-1}	η_1	η_2	η_3	$\cdots$	η_{n-2}	η_{n-1}
1	1	1	1		1	1	1	0	0		0	0
2	1	1	1		1	1	0	1	0		0	0
3	1	1	1		1	1	0	0	1		0	0
$\vdots$												
n-2	1	1	1		1	1	0	0	0		1	0
n-1	1	1	1		1	1	0	0	0		0	1
n	1	1	1		1	1	0	0	0		0	0

bound of C is achieved with equality. Figure 6 shows the multiple experiment tree for this machine and $\mathcal{A} = \{1, 2, \cdots, q\}$.

Instead of applying all the $E_i^{(k)}$ sequences simultaneously, one may apply only those which are judged necessary on the basis of previously observed responses. In terms of the multiple experiment tree, this means that, when the initial state of M is σ_i, only these machines which appear

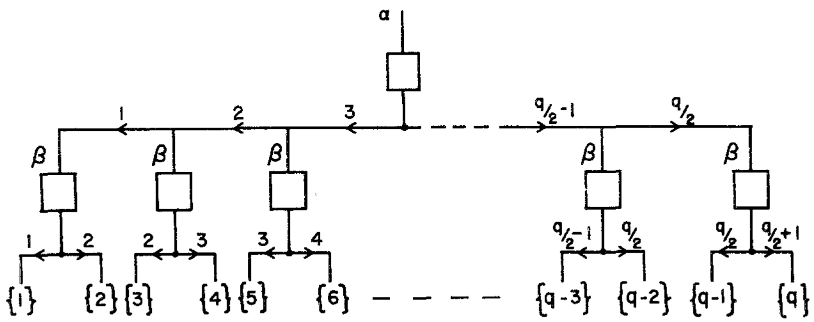


FIG. 7. Multiple experiment tree for machine of Table III

TABLE III

	s_{t+1}		y_t	
s_t \ x_t	α	β	α	β
1	1	1	1	1
2	1	1	1	2
3	1	1	2	2
4	1	1	2	3
5	1	1	3	3
6	1	1	3	4
⋮				
n-3	1	1	$\frac{n}{2}$-1	$\frac{n}{2}$-1
n-2	1	1	$\frac{n}{2}$-1	$\frac{n}{2}$
n-1	1	1	$\frac{n}{2}$	$\frac{n}{2}$
n	1	1	$\frac{n}{2}$	1

on the path terminating in $\{\sigma_i\}$ are utilized. This scheme, which constitutes a multiple adaptive experiment for solving the distinguishing problem, is more economical than the preset version, inasmuch as it eliminates the need for all the copies which do not appear on the traversed path. The maximum number of copies which may be needed in any adaptive experiment cannot exceed the number of levels L in the multiple experiment tree; to the extent that L is lower than C (clearly, L can never exceed C), the adaptive experiment is more advantageous than the preset one. Figure 6 is an example where $L = C$ and the adaptive experiment may not always have an advantage (for example, when state q is the initial state). Table III is the state table of an n-state machine (n even) for which the advantage of an adaptive experiment over a preset one is always relatively great. Figure 7 shows the multiple experiment tree for this machine and $\mathcal{A} = \{1, 2, \cdots, q\}$ ($4 \leqq q \leqq n - 2$ and even); the preset experiment in this case is seen to require $(q/2) + 1$ copies, while the adaptive experiment requires only 2 copies.

As was previously remarked, the design procedure for the multiple experiment does not necessarily minimize the number of required copies C. In some cases this number can be reduced by exploiting the following obvious property: Given two input sequences E_1 and E_1E_2, the response of M to E_1 can be deduced from the response of M to E_1E_2. An example are the sequences α, $\alpha\alpha$, and $\alpha\alpha\alpha$ in the multiple experiment tree of Fig. 5. The responses of A to α and $\alpha\alpha$ can be deduced from the response to $\alpha\alpha\alpha$, so that α and $\alpha\alpha$ need not be applied. This reduces the number of copies required in this problem from 4 to 2; clearly, no further reduction is possible, since the problem is unsolvable by a simple experiment.

V. MINIMAL HOMING EXPERIMENTS

While the distinguishing problem cannot, in general, be solved by a simple experiment, the homing problem can always be solved by a simple experiment, as will be shown in the next section. The tool for designing minimal homing experiments is the *homing tree*, constructed by modifying rules (ii) and (iv) formulated for the successor tree. The new rules, numbered (ii′) and (iv′) respectively, are the following:

(ii′) Let branch "S" be a terminal branch if there is a branch "S" in a previously completed level.

(iv′) Terminate construction when no new branches can be generated, or when any of the S-groups associated with the last completed level is homogeneous.

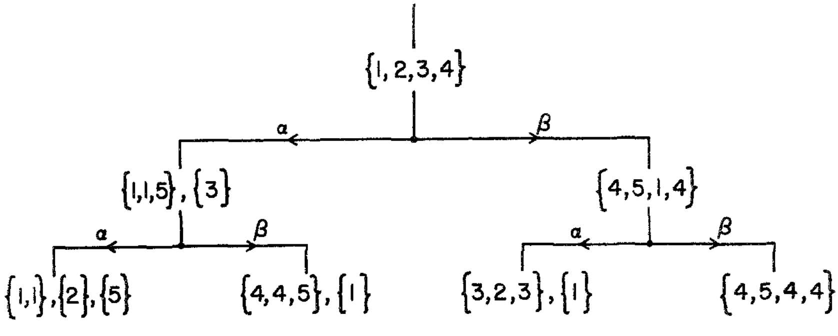


FIG. 8. Homing tree for A and $\mathcal{A} = \{1, 2, 3, 4\}$

Thus, the homing tree is the successor tree with rule (ii), (a) omitted, and with homogeneous groups replacing simple groups as the ones which call for construction termination. As an example, Fig. 8 shows the homing tree for A and $\mathcal{A} = \{1, 2, 3, 4\}$.

A *homing path* will be defined as any path in the homing tree which leads from $\mathcal{A}$ to a branch associated with a homogeneous group. A *homing sequence* for M and $\mathcal{A}$ is any input sequence which, when applied to M, passes it from the admissible state in which it was prior to application into a known final state.

THEOREM 5. *A homing path in a homing tree for machine M and admissible set $\mathcal{A}$ describes a homing sequence for M and $\mathcal{A}$.*

PROOF. Suppose the path describing $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ leads into the S-group $\{S_1\}, \{S_2\}, \cdots, \{S_r\}$. Then the response of M to $\eta_{i_1}\eta_{i_2} \cdots \eta_{i_l}$ uniquely determines, by construction, to which set $\{S_i\}$ the final state of M belongs. If the group is homogeneous, all the states in any $\{S_i\}$ are identical, and knowledge of the set implies knowledge of the final state.

The shortest homing sequence for machine M and admissible set $\mathcal{A}$, will be called the *minimal homing sequence* for M and $\mathcal{A}$.

THEOREM 6. *The set of homing paths in the homing tree for machine M and admissible set $\mathcal{A}$, describes all minimal homing sequences for M and $\mathcal{A}$, and no homing sequences other than minimal.*

PROOF. By the same argument used in the proof of Theorem 3, the inclusion of rule (ii′) cannot affect the enumeration of minimal homing sequences. Thus, the set of homing paths in the homing tree describes all minimal homing sequences for M and $\mathcal{A}$. By virtue of rule (iv′), the homing tree cannot contain any homing paths of distinct lengths, which proves the theorem.

The procedure for solving the homing problem by a minimal, simple, preset experiment can now be outlined as follows:

(1) Construct the homing tree for the given machine M and $\mathcal{A} = \{\sigma_1, \sigma_2, \cdots, \sigma_q\}$.

(2) Choose any of the homing paths exhibited by the tree, say E, and list the responses and final states of $\sigma_1, \sigma_2, \cdots, \sigma_q$ when E is applied to M.

(3) Apply E to M and determine the final state of M by comparing the response with the list compiled in step (2).

As an example, the minimal-homing sequence for A and $\mathcal{A} = \{1, 2, 3, 4\}$ is seen from Fig. 8 to be $\alpha\alpha$ (leading to the homogeneous group $\{1, 1\}$, $\{2\}$, $\{5\}$). $\alpha\alpha$ passes states 1 and 2 into state 1 with the response 00, state 3 into state 2 with the response 01, and state 4 into state 5 with the response 10. The response, then, uniquely determines the final (although not the initial) state of A.

A homing path can be segmented according to the following algorithm: Calling $\mathcal{A}$ "$S^{(1)}$", let the kth segment be the subpath leading from "$S^{(k)}$" to the first S-group, denoted by $S^{(k+1)}$, which contains at least one more homogeneous S_i-set than $S^{(k)}$. Since the number of S_i-sets in the last S-group cannot exceed q, the total number of segments cannot exceed $q - 1$. Denoting the subsequence described by the kth segment by E_k, an adaptive experiment can be conducted according to the following program: Starting with $k = 1$, apply E_k to M and observe the response. If the response is attributable to a subset of $\mathcal{A}$ which, when E_k is applied, passes into a single state of M, this state is the sought final state; if not, increment k by 1 and repeat.

This experiment is seen to be a minimal, simple, adaptive experiment for solving the homing problem. It may terminate after any E_k, depending on the initial state σ_i of M. If $S^{(k+1)}$ contains the homogeneous S_i-set $\{\sigma_i', \sigma_i', \cdots, \sigma_i'\}$, and if applying the input sequence $E_1E_2 \cdots E_k$ passes σ_i into σ_i', then the response of M to $E_1E_2 \cdots E_k$ is uniquely attributable to the final state σ_i' and no additional experimentation is necessary. As was the case with the minimal, simple, adaptive experiment for solving the distinguishing problem, there is always a nonzero a priori probability that the adaptive experiment will be shorter than the preset one.

As an example, the problem of Fig. 8 can be solved by an adaptive experiment, by segmenting the minimal homing sequence $\alpha\alpha$ into α and α. If the initial state of A is 4, the response to the first subsequence

α is 1, which can be attributed to the final state 3 only. The homing experiment, in this case, would terminate after the application of a single input symbol.

VI. REGULAR HOMING EXPERIMENTS

As could be observed in the preceding section, a minimal homing experiment is designed at the cost of constructing a homing tree which, in problems involving large admissible sets, becomes quite cumbersome. If one is content with any finite homing experiment (whose length is subject to a bound to be determined below), simpler design procedures are available.

Consider an "extended" homing tree, constructed according to rules (i) and (iii) only; clearly, this tree describes all homing sequences—minimal and nonminimal. Also, consider a path in the extended tree, composed of consecutive subpaths traced according to the following criterion: If the current branch is associated with an S-group containing exactly d nonhomogeneous S_i-sets, proceed to a branch associated with an S-group containing at least d more S_i-sets than the current one. Assuming that this "tracing criterion" can always be satisfied, it results in a path leading to a homogeneous S-group, and hence in a path describing a homing sequence.

Let $\{S_1\}, \{S_2\}, \cdots, \{S_r\}$ be an S-group in which $\{S_1\}, \{S_2\}, \cdots, \{S_d\}$ $(d \leqq r)$ are the nonhomogeneous S_i-sets, and in which $\{S_i\}$ contains w_i states. $\{S_1\}$ contains at least one pair of states which are distinguishable by an input sequence of length $n - w_1 + 1$ or less; denote the minimal such sequence by E_1. The E_1-successor of S, denoted by $S^{(1)}$, contains, therefore, at least $r + 1$ S_i-sets. If $S^{(1)}$ contains less than $r + d$ S_i-sets, there must be an S_i-set in $S^{(1)}$ which contains the same number of states as some S_i-set in S; let these two S_i-sets be $\{S_2'\}$ and $\{S_2\}$, respectively. $\{S_2'\}$ contains at least one pair of states which are distinguishable by an input sequence of length $n - w_2 + 1$ or less; denote the minimal such sequence by E_2. The E_2-successor of $S^{(1)}$, denoted by $S^{(2)}$, contains, therefore, at least $r + 2$ S_i-sets. In general: If $S^{(k)}$ contains less than $r + d$ S_i-sets, there must be an S_i-set in $S^{(k)}$ which contains the same number of states as some S_i-set in S; let these two S_i-sets be $\{S_{k+1}'\}$ and $\{S_{k+1}\}$, respectively. $\{S_{k+1}'\}$ contains at least one pair of states which are distinguishable by an input sequence of length $n - w_{k+1} + 1$ or less; denote the minimal such sequence by E_{k+1}. The E_{k+1}-successor of $S^{(k)}$, denoted by $S^{(k+1)}$, contains, therefore, at least

$r + k + 1$ S_i-sets. If $S^{(k+1)}$ contains less than $r + d$ S_i-sets, increment k by 1 and repeat. Thus, the tracing ceiterion can always be satisfied by an input sequence $E_1E_2 \cdots E_d$ whose length is at most

$$\sum_{k=1}^{d} (n - w_k + 1) = d(n + 1) - \sum_{k=1}^{d} w_k$$
$$\leqq r(n + 1) - \sum_{k=1}^{r} w_k = r(n + 1) - q.$$

Moreover, the determination of the homing path does not require the actual construction of the extended homing tree, but can be done by consecutively constructing the $E_1E_2 \cdots E_d$ sequences, as described above.

Using the last inequality, the length of the first subpath is seen to be $n + 1 - q$ or less; since this subpath leads to an S-group containing at least 2 S_i-sets, the length of the second subpath is $2(n + 1) - q$ or less. In general, the length of the kth subpath is $2^{k-1}(n + 1) - q$ or less, and it leads to a group containing at least 2^k S_i-sets. Since the number of S_i-sets cannot exceed q, when q is a power of 2 the number of subpaths cannot exceed $\log_2 q$. Hence, the total length of the homing sequence, when q is a power of 2, is at most

$$\sum_{k=1}^{\log_2 q} 2^{k-1}(n + 1) - q = (n + 1)(q - 1) - q \log_2 q$$

When q is not a power of 2, the last expression should be rounded to the lowest integer which exceeds it. We thus have the following theorem:

THEOREM 7. *An n-state machine having an admissible set of size q can always be passed into a known final state by a simple, preset experiment whose length is at most $(n + 1)(q - 1) - q \log_2 \bar{q}$, where $\bar{q}$ is the largest power of 2 not exceeding q.*

The above theorem agrees with a more general formulation derived in Ginsburg's paper.

Table IV illustrates how a regular, simple, preset experiment is constructed for the homing problem of A and $\mathfrak{A} = \{1, 2, 3, 4, 5\}$. The "initial group" and the "final group" represent the S-groups associated with the first and last branches, respectively, of each subpath in the homing path under construction. The "distinguishable pair" is a pair of states in any S_i-set belonging to the initial group; the pair is chosen according to the same criterion used in the design of the adaptive experiment for the dis-

TABLE IV

Initial group	Distinguishable pair	Distinguishing sequence	Final group
$\{1, 2, 3, 4, 5\}$	1, 4	α	$\{1, 1, 5\}$, $\{3, 2\}$
$\{1, 1, 5\}$, $\{3, 2\}$	1, 5	α	$\{1, 1\}$, $\{2\}$, $\{5, 1\}$
$\{1, 1\}$, $\{2\}$, $\{5, 1\}$	1, 5	α	$\{1, 1\}$, $\{1\}$, $\{2\}$, $\{1\}$

TABLE V

Initial state	Response to $\alpha\alpha\alpha$	Final state
1	000	1
2	000	1
3	010	1
4	101	2
5	100	1

tinguishing problem. The total homing sequence is constructed by assembling the distinguishing sequences in the order of their determination. In our example, this sequence is seen to be $\alpha\alpha\alpha$. Table V lists the responses and final states of A, for every admissible state, when the input sequence is $\alpha\alpha\alpha$. This table may be referred to at the end of the experiment to deduce the final state from the observed response.

An adaptive version of the regular homing experiment can be conducted as follows: Apply the first distinguishing sequence designed for the preset experiment. On the basis of the observed response, all but one of the S_i-sets in the first "final group" can be eliminated. The surviving S_i-set, which is necessarily smaller than $\mathfrak{A}$, is denoted by $\mathfrak{A}_1$ and may be regarded as a new admissible set defining a new homing problem for M. In general: Determine the first distinguishing sequence in the preset homing experiment for M and $\mathfrak{A}_k$; on the basis of the observed response, all but one of the S_i-sets in the first "final group" can be eliminated. The surviving S_i-set, which is necessarily smaller than $\mathfrak{A}_k$, is denoted by $\mathfrak{A}_{k+1}$ and may be regarded as a new admissible set defining a new homing problem for M. If $\mathfrak{A}_{k+1}$ contains a single state, this state is the final state sought; if not, increment k by 1 and repeat. The total experiment consists of at most $q - 1$ sequences, and its length, as determined in Ginsburg's paper, is at most $(2n - q)(q - 1)/2$. In Moore's paper, the described procedure is particularized for $q = n$.

As an example, consider the design of a regular, adaptive homing experiment for A and $\mathcal{A} = \{1, 2, 3, 4, 5\}$, where the initial state of A is 4. Applying the first distinguishing sequence of Table IV, α, to A yields the response 1. The surviving S_i-set in the final group is, then, $\{3, 2\}$. The distinguishing sequence for the pair $\{3, 2\}$ is $\alpha\alpha$, which yields the response 01. The surviving S_i-set in the final group is, then, $\{2\}$, which is the final state.

VII. MISCELLANEOUS COROLLARIES

In connection with the distinguishing problem, it is of interest to mention the property of "η_i-mergability" (η_i being a symbol in the input alphabet of M): A machine M is called *η_i-mergable*, if in its state table the η_i entries are identical (in both the s_{t+1} and y_t subtables) in two or more rows.

THEOREM 8. *Let η_i be the ith symbol in the input alphabet of machine M. (a) If M is η_i-mergable for every η_i, then at least one distinguishing problem for M is unsolvable by a simple experiment. (b) If M is not η_i-mergable for any η_i, then all distinguishing problems for M are solvable by a simple experiment.*

PROOF. Let R_i be the set of states corresponding to the rows in which the η_i entries are identical, and let $\bigcup_{i=1}^{m} R_i$, where m is the size of the input alphabet, be the admissible set $\mathcal{A}$. Then, for every input symbol applied to M, there are at least two states in $\mathcal{A}$ which pass into the same state with identical responses. Hence, if the admissible set is as specified above, the distinguishing problem cannot be solved by any simple experiment. (b) Since M is not η_i-mergable for any η_i, the present state, the present input, and the present output uniquely determine the previous state. Hence, if the final state and the input and output sequences which led to it are known, the initial state can always be determined. Since, by Theorem 7, every machine can be passed into a known final state, the theorem follows.

A set of machines $\{M_1, M_2, \cdots, M_l\}$ will be called *exclusive*, if each M_i is a reduced machine, and if no state in M_i is equivalent to any state in $M_j (i \neq j)$. All the results obtained in this paper are directly applicable to the machine $\bigcup_{i=1}^{l} M_i$, where $\{M_1, M_2, \cdots, M_l\}$ is an exclusive set, and where $\bigcup_{i=1}^{l} M_i$ is the machine obtained by combining the state tables of $M_1, M_2, \cdots, M_l$ into a single table. The problem of identifying the initial (or final) state of a given machine M when it is known that M

is some $M_i(i = 1, 2, \cdots, l)$ in a state belonging to the admissible set $\mathcal{A}_i$, is precisely the distinguishing (or homing) problem for $\bigcup_{i=1}^{l} M_i$ and $\mathcal{A} = \bigcup_{i=1}^{l} \mathcal{A}_i$. In particular, the problem of identifying M when it is known that M is some M_i, is precisely the homing problem for $\bigcup_{i=1}^{l} M_i$ and an admissible set which contains all the states of $\bigcup_{i=1}^{l} M_i$. Thus, the solution of the homing problem is also the solution to the problem of identifying an unknown machine out of a known exclusive set.

It should be noted that if the set $\{M_1, M_2, \cdots, M_l\}$ is a set of reduced, strongly connected machines, then it is also an exclusive set; the converse, however, is not necessarily true. Thus, the conclusions reached in the preceding paragraph are more general than those reached in Moore's paper, where their applicability is limited to strongly connected machines only.

VIII. CONCLUSIONS

In this paper we presented general procedures for solving distinguishing and homing problems, thereby filling some of the gaps which existed in the treatment of state-identification in the theory of finite automata. In closing, we shall mention some of the work that remains to be done in this area.

The procedures we proposed for designing minimal simple experiments are based on the concept of the successor tree, and are semienumerational in nature. These procedures could, perhaps, be improved upon by formulating additional rules for the construction of the successor tree, which will force more redundant paths to be eliminated in the early stages of construction. Also desired are procedures for designing regular, simple, preset and adaptive experiments, analogous to those described for the homing problem, which would be suitable for solving the distinguishing problem whenever simple experimentation is realizable. Finally, it would be desirable to have a procedure for designing multiple, preset or adaptive experiments for solving the distinguishing problem (whenever simple experimentation is not realizable), which are minimal either with respect to the number of required copies, or with respect to the length of the total input sequence.

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